SYMMETRY GROUPS AND LAGRANGIANS ASSOCIATED TO ŢIŢEICA SURFACES

N.BÎLĂ

Abstract

Ţiţeica proved [23] that the surfaces for which the ratio $\frac{K}{d^4}$ is constant (where K is the Gaussian curvature and d is the distance from the origin to the tangent plane at an arbitrary point) are invariants under the group of centroaffine transformations. In this paper one applies the symmetry groups theory ([20], [21]) to study the PDEs which arise in Ţiţeica surfaces theory: the PDEs system (5) (equivalent to (15)+(16)) with the particular cases (7) and (9), the PDEs system (11) and the Liouville-Ţiţeica PDE (8), respectively the Ţiţeica PDE (10) (equivalent to (8'), respectively (10')). In the case of the PDEs systems (15)+(16) and (11), the center of our attention is to find the symmetry subgroups G_1 of the full symmetry group G, respectively $\bar{G}_1$ of the full symmetry group $\bar{G}$, which act on the space of dependent variables, and also the symmetry subgroups G_2 of G and $\bar{G}_2$ of $\bar{G}$, which act on the space of independent variables (Theorems 4,5,6 and 7). One proves (Theorem 4) that the subgroup G_1 is the unimodular subgroup of the group of the centroaffine transformations. One gives a new solution (21) for the Ţiţeica PDE for which it is a ruled Ţiţeica surface associated (Proposition 1). One finds the symmetry groups of Liouville-Ţiţeica PDE and Ţiţeica PDE (Theorems 9 and 10) and one proves that these are Euler-Lagrange equations with the Lagrangians (32) and (33) (Theorem 12). One gets the variational symmetry groups of the associated functionals (38) and (39) (Theorems 16 and 17) and conservations laws (Proposition 3). We make the remark that the Ţiţeica simple surfaces $z = f(x, y)$ was studied also in [28] and we proved that the Ţiţeica PDE is an Euler-Lagrange equation. All these results shows that Ţiţeica surfaces theory is strongly related to variational problems and hence it is a subject of global differential geometry.

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1 Introduction

Ţiţeica-the founder of the centroaffine geometry- introduced in 1907 a new class of surfaces, called *the surfaces S*, with the property that $\frac{K}{d^4}$=constant, where K

is the Gaussian curvature and d is the distance from the origin to the tangent plane at an arbitrary point [23]. These were called *Ţiţeica surfaces* by Gheorghiu, or *affine spheres* by Blaschke and *projectives spheres* by Wilczynski. The most simple Ţiţeica surfaces are the spheres and the quadrics. The extension of this class to hypersurfaces was considered by Ţiţeica itself [24],[25]. Also, Mayer [19], Gheorghiu [12], Dobrescu [10] and Vrănceanu [29], [30] studied the properties of these hypersurfaces. Gheorghiu maked a remark on the hypersurfaces Ţiţeica: these can be considered as the affine spaces A_{n-1}, embedded in a affine Euclidean space E_n. Using this result, he introduced a new class of affine space A_n^0 and new examples of these were considered by Udrişte [26]. We start to make a short presentation of the Ţiţeica surfaces.

Let $D \subset \mathbf{R}^2$ be an open set and let

$$\Sigma: \ \ \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \ \ (u,v) \in D,$$

be a surface in $\mathbf{R}^3$, different from a cone with the vertex at the origin of the system of coordinates. Thus, the position vector $\mathbf{r}$ of an arbitrary point of the surface satisfies the condition

$$(1) \qquad (\mathbf{r}, \mathbf{r}_u, \mathbf{r}_v) \neq 0,$$

and this can be considered the solution of the second order PDEs system,

$$(2) \qquad \begin{cases} \mathbf{r}_{uu} &= a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{r} \\ \mathbf{r}_{uv} &= a'\mathbf{r}_u + b'\mathbf{r}_v + c'\mathbf{r} \\ \mathbf{r}_{vv} &= a''\mathbf{r}_u + b''\mathbf{r}_v + c''\mathbf{r}, \end{cases}$$

which is completely integrable, i.e.,

$$(3) \qquad (\mathbf{r}_{uu})_v = (\mathbf{r}_{uv})_u, \quad (\mathbf{r}_{uv})_v = (\mathbf{r}_{vv})_u,$$

where $a, a', a'', ...$ are nine functions of u and v. The above system defines a surface, leaving a centroaffinity aside. The coeficients $a, a', a'', ...$ are called *the centroaffine invariants*. If the surface Σ is related to the asymptotic lines (if a surface is not developable, then the two families of the asymptotic lines are distinct), then $c = c'' = 0$, and thus it is defined by the following completely integrable second order PDEs system

$$(4) \qquad \begin{cases} \mathbf{r}_{uu} &= a\mathbf{r}_u + b\mathbf{r}_v \\ \mathbf{r}_{uv} &= a'\mathbf{r}_u + b'\mathbf{r}_v + c'\mathbf{r} \\ \mathbf{r}_{vv} &= a''\mathbf{r}_u + b''\mathbf{r}_v. \end{cases}$$

Theorem 1 (Ţiţeica). *Let Σ be a surface related to the asymptotic lines. The ratio $I = \frac{K}{d^4}$ is a constant if and only if $a' = b' = 0$.*

Thus, the surfaces Ţiţeica are defined by the PDEs system

$$(5) \qquad \begin{cases} \mathbf{r}_{uu} &= a\mathbf{r}_u + b\mathbf{r}_v \\ \mathbf{r}_{uv} &= h\mathbf{r} \\ \mathbf{r}_{vv} &= a''\mathbf{r}_u + b''\mathbf{r}_v, \end{cases}$$

where we denote $c' = h$, and for which the integrability conditions (3) turn in

$$ah = h_u, \quad a_v = ba'' + h, \quad b_v + bb'' = 0, \tag{6}$$

$$h_v = b''h, \quad a''_u + aa'' = 0, \quad h = b''_u + a''b.$$

Remark. In the particular cases $b = 0$ or $a'' = 0$, Σ is a simply ruled surface. Thus, for $b = 0,\ a'' \neq 0$: the coordinates curves $v = v_0$ are straight lines, and for $b \neq 0,\ a'' = 0$: the coordinates curves $u = u_0$ are straight lines. If $b = a'' = 0$, then Σ is a double ruled surface (a quadric surface).

The ruled Ţiţeica surfaces are given by the PDEs system

$$\begin{cases} \mathbf{r}_{uu} &= \frac{h_u}{h}\mathbf{r}_u + \frac{\varphi(u)}{h}\mathbf{r}_v \\ \mathbf{r}_{uv} &= h\mathbf{r} \\ \mathbf{r}_{vv} &= \frac{h_v}{h}\mathbf{r}_v, \end{cases} \tag{7}$$

where h is a solution of the Liouville-Ţiţeica PDE

$$(\ln h)_{uv} = h. \tag{8}$$

The Ţiţeica surfaces which are not ruled surfaces, are given by the PDEs system

$$\begin{cases} \mathbf{r}_{uu} &= \frac{h_u}{h}\mathbf{r}_u + \frac{1}{h}\mathbf{r}_v \\ \mathbf{r}_{uv} &= h\mathbf{r} \\ \mathbf{r}_{vv} &= \frac{1}{h}\mathbf{r}_u + \frac{h_v}{h}\mathbf{r}_v, \end{cases} \tag{9}$$

where h is a solution of the Ţiţeica PDE

$$(\ln h)_{uv} = h - \frac{1}{h^2}. \tag{10}$$

The PDEs system (5) can be identified with the completely integrable scalar PDEs system

$$\begin{cases} \theta_{uu} &= a\theta_u + b\theta_v \\ \theta_{uv} &= h\theta \\ \theta_{vv} &= a''\theta_u + b''\theta_v, \end{cases} \tag{11}$$

with the condition that three independent solutions of (11)+(6): $x = x(u, v), y = y(u, v), z = z(u, v)$, define a Ţiţeica surface. It is known that every linear combination of x, y, z is a solution of the system (11) also. Thus, a surface Σ is determined leaving a centroaffinity aside.

On the other hand, it is known that Sophus Lie is the founder of theory of symmetry groups. A modern presentation using the jets theory is introduced by Olver in his book [20]. A local group of transformations on the space of the independent and dependent variables of a studied PDEs system which transforms

the solutions of the system into its solutions, is called *symmetry group* or *strong symmetry group* of the system. The symmetry groups theory is very applied to study the ODEs, PDEs systems which appear in Geometry, Mechanics and Physics [2]-[6],[8],[9],[14],[17],[18],[20],[21],[27],[28],[31]. There are many computational programs for finding the defining system of infinitesimal symmetries, but in the ours cases of the PDEs systems (5) and (11) we cannot apply these. We make the remark that a other point of view in the study of the Liouville-Ţiţeica PDE and Ţiţeica PDE is contained the papers of Bobenko [7] and Wolf [31].

In this paper we shall apply this theory for finding infinitesimal symmetries of the PDEs systems which arise in Ţiţeica surfaces theory and we shall give a new point of view of Ţiţeica theory with the connection of the known results. We shall adopt the notation of the book of Olver [20].

2 Symmetry Group of PDEs System

Let consider the PDEs system

$$\Delta_\nu(x, u^{(n)}) = 0, \;\; \nu = 1, ..., l, \tag{12}$$

with $x = (x^1, ..., x^p), \; u = (u^1, ..., u^q)$ and $\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), ..., \Delta_l(x, u^{(n)}))$, is a differentiable function. We note $u^{(n)}$ all the partial derivatives of the function u to 0 up n. Any function $u = h(x)$, $h : D \subset \mathbf{R}^p \to U \subset \mathbf{R}^q$, $\;h = (h^1, ..., h^q)$, induces the function $u^{(n)} = pr^{(n)}h$ called *the n-th prolongation of h*, which is defined by $u^\alpha_J = \partial_J h^\alpha$, $pr^{(n)}h : D \to U^{(n)}$, and for each $x \in D$, $pr^{(n)}h$ is a vector whose $qp^{(n)} = C^n_{p+n}$ entries represent the values of h and all its derivatives up to order n at the point x.

The space $D \times U^{(n)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order n, is called *the n-th order jet space* of the underlying space $D \times U$. Thus Δ is a map from the jet space $D \times U^{(n)}$ to $\mathbf{R}^l$. The PDEs system (12) determine the subvariety

$$\mathcal{S} = \{(x, u^{(n)}) \,|\Delta(x, u^{(n)}) = 0\}$$

of the total jet space $D \times U^{(n)}$. One identifies the system of PDEs (12) with its corresponding subvariety $\mathcal{S}$.

Let $M \subset D \times U$ be an open set. *A symmety group of the PDEs system (12)* is a local group of transformations G acting on M with the property that whenever $u = f(x)$ is a solution of (12) and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system. The system (12) is called *invariant with respect to G*.

Let us consider X a vector field on M with corresponding (local) 1-parameter group $\exp(\varepsilon X)$ which is the infinitesimal generator of the symmetry group of the PDEs system (12). The infinitesimal generator of the corresponding prolonged

1-parameter group $pr^{(n)}[\exp(\varepsilon X)]$:

$$pr^{(n)}X|_{(x,u^{(n)})} = \frac{d}{d\varepsilon} pr^{(n)}[\exp(\varepsilon X)](x, u^{(n)})|_{\varepsilon=0}$$

for any $(x, u^{(n)}) \in M^{(n)}$, is a vector field on the n-jet space $M^{(n)}$ called *the n-th prolongation of* X and denoted by $pr^{(n)}X$.

The PDEs system (12) is called to be of *maximal rank* if the Jacobi matrix

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u^\alpha_J} \right)$$

of Δ, with respect to all the variables $(x, u^{(n)})$,is of rank l whenever $\Delta(x, u^{(n)}) = 0$.

Theorem 2. *Let*

$$X = \sum_{i=1}^{p} \zeta^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

be a vector field on open set $M \subset D \times U$. The n-th prolongation of X is the vector field

$$pr^{(n)}X = X + \sum_{\alpha=1}^{q} \sum_{J} \phi^J_\alpha(x, u^{(n)}) \frac{\partial}{\partial u^\alpha_J}, \tag{13}$$

defined on the corresponding jet space $M^{(n)} \subset D \times U^{(n)}$, the second summation being over all multi-indices $J = (j_1, ... j_k)$ whith $1 \le j_k \le p$, $1 \le k \le n$. The coefficient functions ϕ^J_α of $pr^{(n)}X$ are given by the following formula

$$\phi^J_\alpha(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \zeta^i u^\alpha_i \right) + \sum_{i=1}^{p} \zeta^i u^\alpha_{J,i},$$

where $u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i}$, $u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i}$.

Theorem 3 (Criterion of infinitesimal invariance). *Let us consider the PDEs system (12) of maximal rank defined over $M \subset D \times U$. If G is a local group of transformations acting on M and*

$$pr^{(n)}X[\Delta_\nu(x, u^n)] = 0, \quad \nu = 1, ..., l, \tag{14}$$

whenever $\Delta_\nu(x, u^{(n)}) = 0$, for every infinitesimal generator X of G, then G is a symmetry group of the PDEs system (12).

The algorithm for finding the symmetry group G of the PDEs system (12): One considers the vector field X on M and one writes the infinitesimal invariance condition (14); one eliminates any dependence between partial derivatives of the functions u^α, determined by the PDEs system (12); one writes the condition (14) like polynomials in the partial derivatives of u^α; one equates with zero the coefficients of partial derivatives of u^α in (14); it follows a PDEs system with respect to the unknown functions ζ^i, ϕ_α and this system defines the symmetry group G of the studied PDEs system.

3 Symmetry Groups Associated to PDEs Systems of Ţiţeica Surfaces

3.1. In the first part of this section we shall study the symmetries of the PDEs system (5), which can be considered in the equivalent form

$$(15)\qquad \begin{cases} x_{uu} & = & a x_u + b x_v \\ x_{uv} & = & h x \\ x_{vv} & = & a'' x_u + b'' x_v \\ y_{uu} & = & a y_u + b y_v \\ y_{uv} & = & h y \\ y_{vv} & = & a'' y_u + b'' y_v \\ z_{uu} & = & a z_u + b z_v \\ z_{uv} & = & h z \\ z_{vv} & = & a'' z_u + b'' z_v, \end{cases}$$

with the conditions (1) and (6). The condition (1) can be written as

$$(16)\qquad (y_u z_v - z_u y_v)x - (x_u z_v - x_v z_u)y + (x_u y_v - x_v y_u)z = f,$$

where $f = f(u,v)$ is a nonzero function. We consider the case of the real asymptotic lines. Let $D \times U^{(2)}$ be the second order jet space associated to the PDEs system (15)+(16), whose coordinates represent the independent variables u, v, the dependent variables x, y, z and the derivatives of the dependent variables till the order two. Denote $x^1 = u,\ x^2 = v,\ u^1 = x,\ u^2 = y$ and $u^3 = z$ (in the above section). Let $M \subset D \times U$ be an open set and let

$$X = \zeta \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \phi \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial z}$$

be the infinitesimal generator of the symmetry group G of the PDEs system (15)+(16), where $\zeta, \eta, \phi, \lambda, \psi$ are functions of u, v, x, y and z.

We shall study if there is a subgroup G_1 of the symmetry group G, which acts on the space of the dependent variables x, y, z of the given system. Let suppose that it is, and let Y be its infinitesimal generator. In this case, we must have $\zeta = 0,\ \eta = 0,\ \phi = \phi(x,y,z),\ \lambda = \lambda(x,y,z),\ \psi = \psi(x,y,z)$. By using the relations (13), we get the second prolongation of the vector field Y, which is defined by the next functions

$$\Phi^u = \phi_x x_u + \phi_y y_u + \phi_z z_u,\quad \Phi^v = \phi_x x_v + \phi_y y_v + \phi_z z_v,$$

$$\begin{aligned} \Phi^{uu} &= \phi_{xx} x_u^2 + \phi_{yy} y_u^2 + \phi_{zz} z_u^2 + 2\phi_{xy} x_u y_u + 2\phi_{xz} x_u z_u + 2\phi_{yz} y_u z_u + \\ &+ \phi_x x_{uu} + \phi_y y_{uu} + \phi_z z_{uu}, \end{aligned}$$

$$\begin{aligned} \Phi^{uv} &= \phi_{xx} x_u x_v + \phi_{xy} x_v y_u + \phi_{xz} x_v z_u + \phi_{xy} x_u y_v + \phi_{yy} y_u y_v + \phi_{yz} y_v z_u + \\ &+ \phi_{xz} z_v x_u + \phi_{yz} y_u z_v + \phi_{zz} z_u z_v + \phi_x x_{uv} + \phi_y y_{uv} + \phi_z z_{uv}, \end{aligned}$$

$$\Phi^{vv} = \phi_{xx}x_v^2 + 2\phi_{xy}x_v y_v + 2\phi_{xz}x_v z_v + 2\phi_{yz}y_v z_v + \phi_{yy}y_v^2 + \phi_{zz}z_v^2 + \\ + \phi_x x_{vv} + \phi_y y_{vv} + \phi_z z_{vv},$$

and also the functions $\Lambda^u, \Lambda^v, \Lambda^{uu}, \Lambda^{uv}, \Lambda^{vv}, \Psi^u, \Psi^v, \Psi^{uu}, \Psi^{uv}, \Psi^{vv}$ which are analogously written by substituting ϕ with λ, and respectively ψ. The PDEs system (15)+(16) is of maximal rank. The infinitesimal invariance condition (14) for (15) turns in

$$(17) \qquad \begin{cases} a\Phi^u + b\Phi^v - \Phi^{uu} = 0 \\ h\Phi - \Phi^{uv} = 0 \\ a''\Phi^u + b''\Phi^v - \Phi^{vv} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \end{cases}$$

Let consider the first relation and let substitute the functions Φ^u, Φ^v and Φ^{uu} gives by the above relations. We find

$$a(\phi_x x_u + \phi_y y_u + \phi_z z_u) + b(\phi_x x_v + \phi_y y_v + \phi_z z_v) - \phi_{xx}x_u^2 - 2\phi_{xy}x_u y_u - 2\phi_{xz}x_u z_u - \\ -2\phi_{yz}y_u z_u - \phi_{yy}y_u^2 - \phi_{zz}z_u^2 - \phi_x x_{uu} - \phi_y y_{uu} - \phi_z z_{uu} = 0.$$

We eliminate any dependencies among the derivatives of the x, y, z by substituting

$$x_{uu} = ax_u + bx_v, \;\; y_{uu} = ay_u + by_v, \;\; z_{uu} = az_u + bz_v.$$

Then it results

$$\phi_{xx}x_u^2 + \phi_{yy}y_u^2 + \phi_{zz}z_u^2 + 2\phi_{xy}x_u y_u + 2\phi_{xz}x_u z_u + 2\phi_{yz}y_u z_u = 0.$$

Now we equate the coefficients of the remaining unconstrained partial derivatives of x, y, z to zero, and we get the PDEs system

$$\phi_{xx} = 0 \quad \phi_{yy} = 0 \quad \phi_{zz} = 0 \quad \phi_{xy} = 0 \quad \phi_{yz} = 0 \quad \phi_{xz} = 0.$$

It follows the solution $\phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z + k$, with $a_{11}, a_{12}, a_{13}, k \in \mathbf{R}$. By substituting the function ϕ in the next two relations of the system (17), we find $k = 0$ and thus $\phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z,\; a_{11}, a_{12}, a_{13} \in \mathbf{R}$. Analogously, using the next six relations of the system (17) we get $\lambda(x, y, z) = a_{21}x + a_{22}y + a_{23}z,\; a_{21}, a_{22}, a_{23} \in \mathbf{R},\;\; \psi(x, y, z) = a_{31}x + a_{32}y + a_{33}z,\; a_{31}, a_{12}, a_{13} \in \mathbf{R}$. Using the criterion of infinitesimal invariance (14), for (16), we get

$$\phi(y_u z_v - z_u y_v) + \lambda(x_v z_u - x_u z_v) + \psi(x_u y_v - x_v y_u) + \Phi^u(zy_v - yz_v) + \Phi^v(yz_u - zy_u) + \\ + \Lambda^u(xz_v - zx_v) + \Lambda^v(zx_u - xz_u) + \Psi^u(yx_v - xy_v) + \Psi^v(xy_u - yx_u) = 0.$$

If we substitute the functions $\Phi^u, \Phi^v, ...$, then it results

$$(x_u y_v - x_v y_u)(\psi - x\psi_x - y\psi_y + z\phi_x + z\lambda_y) + (x_v z_u - x_u z_v)(\lambda - x\lambda_x - z\lambda_z + y\phi_x + \\ + y\psi_z) + (y_u z_v - y_v z_u)(\phi - y\phi_y - z\phi_z + x\lambda_y + x\psi_z) = 0.$$

We eliminate any dependencies among the derivatives of x, y, z by using the relation (16) itself, and we find $\phi_x + \lambda_y + \psi_z = 0$ or equivalent $a_{33} + a_{11} + a_{22} = 0$. Thus, the next functions

$$\begin{array}{rcl} \phi(x,y,z) & = & a_{11}x + a_{12}y + a_{13}z \\ \lambda(x,y,z) & = & a_{21}x + a_{22}y + a_{23}z \\ \psi(x,y,z) & = & a_{31}x + a_{32}y - (a_{11} + a_{22})z \end{array}$$

define the infinitesimal generator Y of the symmetry subgroup G_1:

$$Y = a_{11}\left(x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right) + a_{22}\left(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right) + a_{12}y\frac{\partial}{\partial x} + a_{13}z\frac{\partial}{\partial x} +$$

$$+ a_{21}x\frac{\partial}{\partial y} + a_{23}z\frac{\partial}{\partial y} + a_{31}x\frac{\partial}{\partial z} + a_{32}y\frac{\partial}{\partial z}.$$

Theorem 4. *The Lie algebra associated to the subgroup G_1 of the full symmetry group G of the PDEs system (15)+(16) (G_1 acts on the space of the dependent variables) is generated by the vector fields*

$$Y_1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z},\ \ Y_2 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},\ \ Y_3 = y\frac{\partial}{\partial x},\ \ Y_4 = z\frac{\partial}{\partial x} \tag{18}$$

$$Y_5 = x\frac{\partial}{\partial y},\ \ Y_6 = z\frac{\partial}{\partial y},\ \ Y_7 = x\frac{\partial}{\partial z},\ \ Y_8 = y\frac{\partial}{\partial z},$$

and, thus the Lie subgroup G_1 is the unimodular subgroup of the group of centroaffine transformations.

Using this result, we can find group-invariant solutions of the PDEs system (15)+(16). For example, if we consider the subalgebra described by the vector fields Y_1 and Y_2, then the function F which is invariant under the associated group, satisfies $Y_1(F) = 0$, and $Y_2(F) = 0$. It results $F = \varphi(u, v, xyz)$ and the group-invariant solutions are defined by $u = C_1,\ v = C_2$ and $xyz = C_3$. Thus one gets the known Ţiţeica surfaces

$$z = \frac{C}{xy},\ \ C \in \mathbf{R}. \tag{19}$$

3.2. We shall study if there is a subgroup G_2 of the symmetry group G of the PDEs system (15)+(16) which acts on the space of the independent variables u, v of the system. Let suppose that it is and let Z be its infinitesimal generator. The vector field Z is defined by the functions

$$\zeta = \zeta(u,v),\ \ \eta = \eta(u,v),\ \ \phi = 0,\ \ \lambda = 0,\ \ \psi = 0.$$

In this case, by using the above algorithm for finding the associated symmetries, we get

Theorem 5. *The general vector field of the algebra of the infinitesimal symmetries associated to the subgroup G_2, where G_2 is the subgroup of the full symmetry group G of the PDEs system (15), which acts on the space of the independent variables, is*

$$Z = \zeta(u)\frac{\partial}{\partial u} + \eta(v)\frac{\partial}{\partial v},$$

where the functions ζ and η satisfy the next PDEs system:

$$\left\{\begin{array}{l} \zeta a_u + \eta a_v + a\zeta_u + \zeta_{uu} = 0 \\ \zeta b_u + \eta b_v - b\eta_v + 2b\zeta_u = 0 \\ \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\ \zeta a''_u + \eta a''_v - a''\zeta_u + 2a''\eta_v = 0 \\ \zeta b''_u + \eta b''_v + b''\eta_v + \eta_{vv} = 0, \end{array}\right. \tag{20}$$

and the functions a, b, h, a'', b'' satisfy the integrability conditions (6).

One considers the cases:

1. If Σ is a ruled Ţiţeica surface (7), then the completely integrability conditions (6) are

$$a = \frac{h_u}{h}, \quad b = \frac{\varphi(u)}{h}, \quad a'' = 0, \quad b'' = \frac{h_v}{h},$$

where h is a solution of the Liouville-Ţiţeica PDE (8). In this case, the relations (20) turn in

$$\left\{\begin{array}{c} \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\ \zeta^3 = \frac{k}{\varphi}, \\ hh_{uv} - h_u h_v = h^3. \end{array}\right.$$

Let us consider the change of variables $\zeta = \frac{1}{U'}$ and $\eta = -\frac{1}{V'}$, where $U = U(u)$ and $V = V(v)$. Then the first PDE implies $h = U'V'\mu(U+V)$ and by substituting this function in the last PDE of the above system (Liouville-Ţiţeica PDE), we find the following ODE

$$\mu\mu'' - \mu'^2 = \mu^3,$$

for which

$$\mu(t) = \left\{\begin{array}{ll} \frac{2}{(t+C)^2}, & k = 0 \\ \\ \frac{l^2}{2\cos^2(\frac{l}{2}t+C),} & k = -l^2 \\ \\ \frac{l^2}{2\mathrm{sh}^2(\frac{l}{2}t+C)}, & k = l^2, \ \ l > 0. \end{array}\right.$$

is the general solution, with $t = U + V$. One substitutes in $h = U'V'\mu(U+V)$ and one consider the special change of the functions $\tilde{U} = F(U)$, $\tilde{V} = G(V)$, $\tilde{U} = U + C$, $\tilde{V} = V$: for $k = 0$, $\tilde{U} = \mathrm{th}\frac{l}{2}(U + C)$, $\tilde{V} = \mathrm{th}\frac{l}{2}V$, for $k = l^2$ and

$\tilde{U} = \mathrm{ctg}(\frac{l}{2}U + C)$, $\tilde{V} = \mathrm{tg}\frac{l}{2}V$, for $k = -l^2$, it results the general solution of the Liouville-Ţiţeica ([15],[24]) namely,

$$h(u, v) = \frac{2\tilde{U}'\tilde{V}'}{(\tilde{U} + \tilde{V})^2}.$$

2. If Σ is a Ţiţeica surface which are not ruled surface (9), then the completely integrability conditions (6) turn in

$$a = \frac{h_u}{h}, \ \ b = a'' = \frac{1}{h}, \ \ b'' = \frac{h_v}{h},$$

where h is a solution of the Ţiţeica PDE (10). If we substitute these functions in the system (20), then we get

$$\zeta_u = 0, \quad \eta_v = 0, \quad \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0.$$

It results $\zeta = C_1$, $\eta = C_2$ and $h = \mu(C_1 v - C_2 u)$ and thus

$$Z = C_1 \frac{\partial}{\partial u} + C_2 \frac{\partial}{\partial v}.$$

Let consider the Ţiţeica PDE (10) and let substitute the above function $h = \mu(C_1 v - C_2 u)$. We get the following ODE:

$$-C_1 C_2 (\mu\mu'' - \mu'^2) = \mu^3 - 1.$$

a. If $C_1 C_2 = 0$, then $\mu = 1$ and $h = 1$. This is the Ţiţeica solution [24].
b. If $C_1 C_2 \neq 0$, then denote $k = -\frac{1}{C_1 C_2}$. The above ODE turns in

$$\mu\mu'' - \mu'^2 = k(\mu^3 - 1).$$

We can consider $k = 1$. This ODE can be reduced to the following

$$\mu'^2 = 2\mu^3 + C\mu^2 + 1, \ \ C \in \mathbf{R},$$

and using the change of function $\mu = \frac{1}{2}g$, this becomes

$$g'^2 = g^3 + Cg^2 + 4.$$

Let λ the real solution of the right side polynom of the above ODE. It results that $\lambda \neq 0$, λ is not a triple solution of this and the ODE can writen as

$$g'^2 = (g - \lambda)\left(g^2 - \frac{4}{\lambda^2}g - \frac{4}{\lambda}\right).$$

2.1. If $\lambda = -1$, then $C = -3$ and the ODE is

$$g'^2 = (g + 1)(g - 2)^2.$$

If we consider $g = \frac{1}{w^2} + 2$ than we have the ODE

$$w'^2 = \frac{1}{4}\left(3w^2 + 1\right),$$

for which the solution $w = w(t),\ t = u + v$ is

$$w(t) = \frac{1}{\sqrt{3}}\mathrm{sh}\left(\frac{t\sqrt{3}}{2} + C_1\right),\quad C_1 \in \mathbf{R}.$$

We get the next solution of the Ţiţeica PDE: $h = \frac{1}{2w^2} + 1$ which in the case $C_1 = 0$, it is

$$h(t) = \frac{3}{2\mathrm{sh}^2\left(\frac{t\sqrt{3}}{2}\right)} + 1, \quad t = u + v. \tag{21}$$

2.2. If $\lambda \neq -1$, then the right side polynom has three distinct real solutions ($\lambda > -1$ or $C < -3$) and repectively one is real and two complex ($\lambda < -1$ or $C > -3$). In this case, the integral

$$J = \int \frac{dg}{\sqrt{(g - \lambda)\left(g^2 - \frac{4}{\lambda^2}g - \frac{4}{\lambda}\right)}}$$

can be reduced to a first genus elliptical integral [11]

$$J = \int \frac{d\varphi}{\sqrt{1 - k^2 \sin\varphi}}.$$

We get: for $C \neq -3$, the solutions of the PDE Ţiţeica with the form $h = \mu(u + v)$ are gives in the terms of elliptical functions.

Proposition 1. *The solution (21) gives a revolution Ţiţeica surface. Moreover, it is an associated ruled Ţiţeica surface.*

Proof. În [24], 164-174, Ţiţeica studied the revolution surfaces defined by the system (11) and make the remark that, in this case, the function h must satisfy $h_u = h_v$, and thus $h = \mu(u + v)$. Ţiţeica obtained the above ODE

$$\mu\mu'' - \mu'^2 = \mu^3 - 1,$$

and not integrate it. He proved: by using

$$\frac{\mu'^2 - 2\mu^3 - 1}{4\mu^2} = -k^2,$$

one finds the solution of the studied system: for $k \neq 0$:

$$\theta(u, v) = k_1 e^{\int \frac{h'-1}{2h} d\alpha} \cos k\beta + k_2 e^{\int \frac{h'-1}{2h} d\alpha} \sin k\beta + k_3 e^{\int \frac{h^2}{h'+1} d\alpha}, \tag{22}$$

and for $k = 0$:

$$\theta(u,v) = e^{\int \frac{h'-1}{2h}d\alpha}\left[k_1\left(\beta^2 + \int \frac{4\mu}{\mu'+1}d\alpha\right) + k_2\beta + k_3\right],$$

where $\alpha = u + v,\ \beta = u - v$ şi $k_1, k_2, k_3 \in \mathbf{R}$.

On the other hand, ours calculus imply that $k^2 = -\frac{C}{4}$, and thus the function (21) defines a revolution surface (22). If we consider

$$\tilde{U} = \text{th}\frac{\sqrt{3}}{2}(U + C_1),\quad \tilde{V} = \text{th}\frac{\sqrt{3}}{2}V,$$

it results that the function h can be written in the next form

$$h(u,v) = \frac{2\tilde{U}'\tilde{V}'}{(\tilde{U}+\tilde{V})^2} + 1 = H(u,v) + 1.$$

But, the above calculus, implies that the function H is a solution of Liouville-Ţiţeica PDE (8) and this defines a ruled Ţiţeica surface.

Proposion 2. *The solution of Ţiţeica (22) is invariant under the transformations subgroup of G_2, for which the Lie algebra is generated (in the case of not ruled Ţiţeica surface) by*

$$Z = C_1\frac{\partial}{\partial u} + C_2\frac{\partial}{\partial v}.$$

3.3. In this part, we shall apply the symmetry group theory for the PDEs system (11) with the integrability conditions (6). Let $D \times \bar{U}^{(2)}$ be the second order jet space associated to the PDEs system (11), whose coordinates are the independent variables u, v, the dependent variable θ and the derivatives of the dependent variable till the order two. Denote by $x^1 = u,\ x^2 = v$ and $u^1 = \theta$ (in the second section). Let consider $\bar{M} \subset D \times \bar{U}$ an open set and let

$$\bar{X} = \zeta\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v} + \alpha\frac{\partial}{\partial \theta}$$

be the infinitesimal generator of the symmetry group $\bar{G}$ of the PDEs system (11), where $\zeta,\ \eta$ and α are functions of $u,\ v$ and θ. The relations (13) imply the first and the second prolongations of the vector field $\bar{X}$

$$pr^{(1)}\bar{X} = \bar{X} + \alpha^u\frac{\partial}{\partial \theta_u} + \alpha^v\frac{\partial}{\partial \theta_v},$$

$$pr^{(2)}\bar{X} = pr^{(1)}\bar{X} + \alpha^{uu}\frac{\partial}{\partial \theta_{uu}} + \alpha^{uv}\frac{\partial}{\partial \theta_{uv}} + \alpha^{vv}\frac{\partial}{\partial \theta_{vv}},$$

where

$$\begin{aligned}\alpha^u &= D_u(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uu} + \eta\theta_{uv},\\ \alpha^v &= D_v(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uv} + \eta\theta_{vv},\end{aligned}$$

$$
\begin{aligned}
\alpha^{uu} &= D_{uu}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uuu} + \eta\theta_{uuv},\\
\alpha^{uv} &= D_{uv}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uuv} + \eta\theta_{uvv},\\
\alpha^{vv} &= D_{vv}(\alpha - \zeta\theta_u - \eta\theta_v) + \zeta\theta_{uvv} + \eta\theta_{vvv}.
\end{aligned}
$$

We shall study if there is a subgroup $\bar{G}_1$ of the symmetry group $\bar{G}$, which acts on the space of the dependent variable θ. Let suppose that it is and let

$$
\bar{Y} = \alpha\frac{\partial}{\partial\theta}, \ \ \alpha = \alpha(\theta),
$$

be its infinitesimal generator. In this case, the above algorithm implies

Theorem 6. *The Lie algebra of the infinitesimal symmetries associated to the subgroup $\bar{G}_1$ of the full symmetry group $\bar{G}$ of the PDEs system (11) ($\bar{G}_1$ acts on the space of the dependent variable θ), is generated by the vector field*

$$
\bar{Y}_1 = \theta\frac{\partial}{\partial\theta}. \tag{23}
$$

Analogously, we study the subgroup $\bar{G}_2$ of the symmetry group $\bar{G}$, which acts on the space of the independent variables u, v of the PDEs system (11). Let

$$
\bar{Z} = \zeta\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}, \ \ \zeta = \zeta(u,v), \ \ \eta = \eta(u,v),
$$

be the infinitesimal generator of it. It results

Theorem 7. *The general vector field of the algebra of the infinitesimal symmetries associated to the subgroup $\bar{G}_2$ of the symmetry group $\bar{G}$ of the PDEs system (11) ($\bar{G}_2$ acts on the space of the independent variables u, v), is*

$$
\bar{Z} = \zeta(u)\frac{\partial}{\partial u} + \eta(v)\frac{\partial}{\partial v}, \tag{24}
$$

where ζ and η satisfy the relations (20).

Remark. The subgroup G_2 and $\bar{G}_2$ are the same actions on the space of the independent variables u, v.

3.4. Now we shall study the symmetries of the Liouville-Ţiţeica PDE (8) and Ţiţeica PDE (10). Let consider the PDEs

$$
\omega_{uv} = e^{\omega}, \tag{8$'$}
$$

and respectively

$$
\omega_{uv} = e^{\omega} - e^{-2\omega}, \tag{10$'$}
$$

which are equivalent to Liouville-Ţiţeica PDE, and respectively Ţiţeica PDE, where $\ln h = \omega$. We remark that these PDEs belong of the next class of second order PDE, of maximal rank,

$$
\omega_{uv} = H(\omega), \tag{25}
$$

which was studied by Sophus Lie himself. Also Pucci, Saccomandi, Mansfield have considered such equations. One proves [18] the next result

Theorem 8. *If $\zeta = \zeta(u, v, \omega)$, $\eta = \eta(u, v, \omega)$ and $\phi = \phi(u, v, \omega)$ are the solutions of the PDEs system*

$$\zeta_v = 0 \quad \zeta_\omega = 0 \quad \eta_u = 0 \quad \eta_\omega = 0 \quad \phi_{\omega\omega} = 0 \quad \phi_{u\omega} = 0 \quad \phi_{v\omega} = 0 \tag{26}$$

$$\phi_{uv} + (\phi_\omega - \zeta_u - \eta_v - \phi)H - H'\phi = 0,$$

where $H = H(\omega)$, then

$$X = \zeta\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v} + \phi\frac{\partial}{\partial \omega}$$

is the infinitesimal generator of the symmetry group associated to one PDE of the form (25).

In the case of PDEs (8') and (10'), we get the results

Theorem 9. *The general vector field which describes the algebra of infinitesimal symmetries associated to the Liouville-Ţiţeica PDE (8') is the following*

$$W = f\frac{\partial}{\partial u} + g\frac{\partial}{\partial v} - (f' + g')\frac{\partial}{\partial \omega}, \tag{27}$$

where $f = f(u)$ and $g = g(v)$.

Theorem 10. *The vector fields which generate the Lie algebra of infinitesimal symmetries associated with the PDE Ţiţeica (10') are*

$$U_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \;\; U_2 = \frac{\partial}{\partial u}, \;\; U_3 = \frac{\partial}{\partial v}. \tag{28}$$

Remark. If $\omega = f(u, v)$ is o solution of the Ţiţeica PDE (8'), then the following functions

$$\omega^{(1)} = f(e^\varepsilon u, e^{-\varepsilon}v), \;\; \omega^{(2)} = f(u - \varepsilon, v), \;\; \omega^{(3)} = f(u, v - \varepsilon),$$

where ε is a real number, are also solutions of the equation.

The adjoint representation of the symmetry group of the Ţiţeica PDE (8'), is given by the next table

Table 1

Ad	U_1	U_2	U_3
U_1	U_1	$e^\varepsilon U_2$	$e^{-\varepsilon}U_3$
U_2	$U_1 - \varepsilon U_2$	U_2	U_3
U_3	$U_1 + \varepsilon U_3$	U_2	U_3

The 1-dimensional subalgebras, described by U_2, U_3, $U_2 - U_3$, implies the finding the following group-invariant solutions:

1. For U_2, and respectively U_3, the solution is $\omega = 0$. Thus it results the Ţiţeica solution $h = 1$ of the PDE (8).

2. In the case of the vector field $U_2 - U_3$, the group-invariant solutions have the form $\omega = f(u+v)$ (respectively $h = \mu(u+v)$ for PDE (10)). But this case was considered in the above section.

Now we look for the converse of the Theorem 10:

Theorem 11. *The second order PDE invariant with respect to the symmetry group (28) of Ţiţeica PDE, has the form*

$$H(\omega, \omega_u\omega_v, \omega_{uv}, \omega_{uu}\omega_{vv}) = 0. \tag{29}$$

Proof. One considers the maximal chain of Lie subalgebras of the Lie algebra associated with the studied group:

$$\{U_2\} \subset \{U_2, U_3\} \subset \{U_1, U_2, U_3\}.$$

and a second order PDE: $F(u, v, \omega^{(2)}) = 0$ which is invariant with resect this group. So it satisfies the criterion of infinitesimal invariance for each second prolongation of these vectors fields.

1. For U_2, it results $pr^{(2)}U_2(F) = 0$, and thus $F = F_1(v, \omega^{(2)})$.
2. For U_3, the condition $pr^{(2)}U_3(F) = 0$, implies $F = F_2(\omega^{(2)})$.
3. For U_1, the relation $pr^{(2)}U_1(F) = 0$ is equivalent to

$$U_1(F_2) - \omega_u\frac{\partial F_2}{\partial \omega_u} + \omega_v\frac{\partial F_2}{\partial \omega_v} - 2\omega_{uu}\frac{\partial F_2}{\partial F_2\omega_{uu}} + 2\omega_{vv}\frac{\partial F_2}{\partial \omega_{vv}} = 0,$$

and finally, we get $F_2 = H$ in (29).

4 Lagrangians Associated to Ţiţeica PDEs

4.1. Our present study is for the inverse problem for the PDEs (8') and (10'). We recall that, the simple form of the inverse problem for a PDE in the calculus of variations is to determine if this is identically to an Euler-Lagrange PDE [1],[8],[16],[20],[21], [28],[31].

Let us consider the second order PDE

$$\Delta(u, v, \omega^{(2)}) = 0, \tag{30}$$

where $\omega^{(2)}$ is the second prolongation of the unknown function $\omega = \omega(u, v)$. The PDE (30) is *identically to an Euler-Lagrange equation* if and only if the inegrability Helmholtz conditions

$$\begin{cases} \frac{\partial \Delta}{\partial \omega_u} & = & D_u\left(\frac{\partial \Delta}{\partial \omega_{uu}}\right) + D_v\left(\frac{1}{2}\frac{\partial \Delta}{\partial \omega_{uv}}\right) \\ \frac{\partial T}{\partial \omega_v} & = & D_u\left(\frac{1}{2}\frac{\partial \Delta}{\partial \omega_{uv}}\right) + D_v\left(\frac{\partial \Delta}{\partial \omega_{vv}}\right), \end{cases} \tag{31}$$

are satisfied. In this case, there exists a function L, called *Lagrangian*, such that the Euler-Lagrange PDE

$$E(L) = \frac{\partial L}{\partial \omega} - D_u\left(\frac{\partial L}{\partial \omega_u}\right) - D_v\left(\frac{\partial L}{\partial \omega_v}\right) = 0$$

is equivalent to the PDE (30), in the sense that every solution of the equation (30) is a solution of the Euler-Lagrange equation $E(L) = 0$ and conversely.

Also, the PDE (30) is called *equivalent to an Euler-Lagrange equation* if there exists a nonzero function $f = f(u, v, \omega, \omega_u, \omega_v)$ such that $f \cdot \Delta = E(L)$. The function f is called *variational integrant factor.*

Theorem 12. *The Liouville-Ţiţeica PDE (8') and Ţiţeica PDE (10') are Euler-Lagrange equations with the next associated Lagrangians*

$$L_1(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega}, \tag{32}$$

and

$$L_2(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega} - \frac{1}{2}e^{-2\omega}. \tag{33}$$

Proof. Indeed, one verifies that the Helmholtz integrability conditions (31) are satisfied and one verifies that L_1 and L_2 can be considered associated Lagrangians.

Remark. The PDEs (8) and (10) are equivalent to Euler-Lagrange PDEs, with the variational integrant factor $\frac{1}{h^3}$.

4.2.. We make a short presentation of the theory of variational symmetry groups for the functionals

$$\mathcal{L}[\omega] = \int\int_{\Omega_0} L(u, v, \omega^{(1)})dudv, \tag{34}$$

with Ω_0 is a domain in $\mathbf{R}^2$ [20],[21].

Let $D \subset \Omega_0$ be a subdomain, U an open set in $\mathbf{R}$ and $M \subset D \times U$ an open set. We consider $\omega \in C^2(D),\ \omega = f(u, v)$ such that $\Gamma_\omega = \{(u, v, \omega(u, v)) | (u, v) \in D\} \subset M$. A local group G of transformations on M is called *variational symmetry group for the functional* (34), if $g_\varepsilon \in G,\ g_\varepsilon(u, v, \omega) = (\bar{u}, \bar{v}, \bar{\omega})$, then the function $\bar{\omega} = \bar{f}(\bar{u}, \bar{v}) = (g \cdot f)(\bar{u}, \bar{v})$ is defined on $\bar{\Omega} \subset \Omega_0$ and

$$\int\int_{\bar{D}} L(\bar{u}, \bar{v}, pr^{(1)}\bar{f}(\bar{u}, \bar{v}))d\bar{u}d\bar{v} = \int\int_D L(u, v, pr^{(1)}f(u, v))dudv.$$

Theorem 13 (Infinitesimal criterion for the variational problem). *A connected group G of transformations acting on $M \subset \Omega_0 \times U$ is a group of variational symmetries for the functional (34) if and only if*

$$pr^{(1)}X(L) + L\ Div\xi = 0, \tag{35}$$

is satisfied for $\forall(u, v, \omega^{(2)}) \in M^{(2)} \subset D \times U^{(2)}$ and for any infinitesimal generator

$$X = \zeta(u, v, \omega)\frac{\partial}{\partial u} + \eta(u, v, \omega)\frac{\partial}{\partial v} + \phi(u, v, \omega)\frac{\partial}{\partial \omega}$$

of G, where $\xi = (\zeta, \eta)$ and $Div\xi = D_u\zeta + D_v\eta$.

Theorem 14. *If G is a variational symmetry group of the functional (34), then G is a symmetry group of Euler-Lagrange equation $E(L) = 0$.*

The converse of Theorem 14 is generally false.

Let us consider the PDE (30). *A conservation law* is a divergence expression $Div\ P = 0$ wich vanishes for all solutions $u = f(x)$ of the given PDE. Here $P = (P^1, P^2)$ with $Div\ P = D_uP^1 + D_vP^2$, *the total divergence.* The function P^1 is called *flow associated* and P^2 is called *conserved density to the conservation law.* It results that there exists a function Q such that

$$Div\ P = Q \cdot \Delta. \tag{36}$$

This relation is called *the characteristic form of the conservation law*, and Q is called *the characteristic of the conservation law.*

Let

$$X = \zeta(u, v, \omega)\frac{\partial}{\partial u} + \eta(u, v, \omega)\frac{\partial}{\partial v} + \phi(u, v, \omega)\frac{\partial}{\partial \omega}$$

be a vector field on M. The vector field

$$X_Q = Q\frac{\partial}{\partial u}, \ \ Q = \phi - \zeta\omega_u - \eta\omega_v,$$

is called *vector field of evolution associated to X*, and Q is called *the characteristic associated to X*.

Theorem 15 (Noether Theorem). *Let G be a local Lie group of transformations, which is a symmetry group of the variational problem (34) and let*

$$X = \zeta(u, v, \omega)\frac{\partial}{\partial u} + \eta(u, v, \omega)\frac{\partial}{\partial v} + \phi(u, v, \omega)\frac{\partial}{\partial \omega}$$

be the infinitesimal generator of G. The characteristic Q of the field X is also a characteristic of the conservation law for the associated Euler-Lagrange equation $E(L) = 0$.

One proves ([20], 356) that for the Lagrangian $L = L(u, v, \omega^{(1)})$ we have

$$P = -(A + L\xi) = -(A^1 + L\zeta, A^2 + L\eta) = (P^1, P^2), \ \ A = (A^1, A^2), \tag{37}$$

where $A^1 = Q \cdot E^{(u)}(L)$, $\quad A^2 = Q \cdot E^{(v)}(L)$. The operators $E^{(u)}(L) = \frac{\partial L}{\partial \omega_u}$ and $E^{(v)}(L) = \frac{\partial L}{\partial \omega_v}$ are called *first order Euler operators.*

4.3. Let us consider the first order Lagrangians (32) and (33) and the associated functionals

$$\mathcal{L}[\omega] = \int\int_D L_1(u, v, \omega^{(1)})dudv \tag{38}$$

and

$$\bar{\mathcal{L}}[\omega] = \int\int_D L_2(u, v, \omega^{(1)})dudv, \tag{39}$$

with D is a domain in $\mathbf{R}^2$ and $\omega \in C^2(D)$.

Theorem 16. *The Lie algebras of the variational symmetry groups of the functional (38) is described by the vector fields*

$$(40) \qquad W_1 = u\frac{\partial}{\partial u} - \frac{\partial}{\partial \omega}, \quad W_2 = v\frac{\partial}{\partial v} - \frac{\partial}{\partial \omega}, \quad W_3 = \frac{\partial}{\partial u}, \quad W_4 = \frac{\partial}{\partial v},$$

Proof. According with Theorem 14, the vector fields which determine the Lie algebra of the variational symmetry group are founded between the vector fields of the Lie algebra of the symmetry group of the associated Euler-Lagrange equation. The condition (35) must be verified only for the vector fields in the algebra of the symmetry group of PDE ($8'$). Let us consider the vector field

$$W = f\frac{\partial}{\partial u} + g\frac{\partial}{\partial v} - (f' + g')\frac{\partial}{\partial \omega}, \quad f = f(u), \; g = g(v),$$

which is given by the relation (27) and let

$$pr^{(2)}W = W - (f'' + f'\omega_u)\,\frac{\partial}{\partial \omega_u} - (g'' + g'\omega_v)\,\frac{\partial}{\partial \omega_v},$$

be the second prolongation. Introducing $\xi = (f, g)$ and $Div\xi = f' + g'$ in the relation (35), this turns in $f''\omega_v + g''\omega_u = 0$ and it implies $f'' = g'' = 0$. Thus, we get $f = C_1 u + C_3, \; g = C_2 v + C_4$ and also

$$W = (C_1 u + C_3)\frac{\partial}{\partial u} + (C_2 v + C_4)\frac{\partial}{\partial v} - (C_1 + C_2)\frac{\partial}{\partial \omega} =$$

$$= C_1\left(u\frac{\partial}{\partial u} - \frac{\partial}{\partial \omega}\right) + C_2\left(v\frac{\partial}{\partial v} - \frac{\partial}{\partial \omega}\right) + C_3\frac{\partial}{\partial u} + C_4\frac{\partial}{\partial v}.$$

Theorem 17. *The Lie algebras of the variational symmetry groups of the functional (39) is described by the vector fields*

$$(41) \qquad U_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad U_2 = \frac{\partial}{\partial u} \quad U_3 = \frac{\partial}{\partial v}.$$

Proposition 3. *The associated flows and respectively the conserved density in the case of Liouville-Ţiţeica PDE (8') and respectively for Ţiţeica PDE (10') are*

Table 2

$-W_i$	P^1	P^2
$-W_1$	$\frac{1}{2}\omega_v - ue^{\omega}$	$\frac{1}{2}\omega_u(1 + u\omega_u)$
$-W_2$	$\frac{1}{2}\omega_v(1 + v\omega_v)$	$\frac{1}{2}\omega_u - ve^{\omega}$
$-W_3$	$-e^{\omega}$	$\frac{1}{2}\omega_u^2$
$-W_4$	$\frac{1}{2}\omega_v^2$	$-e^{\omega}$

Table 3

$-U_i$	P^1	P^2
$-U_1$	$-\frac{1}{2}ue^{-2\omega}-\frac{1}{2}v\omega_v^2-ue^{\omega}$	$\frac{1}{2}u\omega_u^2+ve^{\omega}+\frac{1}{2}ve^{-2\omega}$
$-U_2$	$-e^{\omega}-\frac{1}{2}e^{-2\omega}$	$\frac{1}{2}\omega_u^2$
$-U_3$	$\frac{1}{2}\omega_v^2$	$-e^{\omega}-\frac{1}{2}e^{-2\omega}$

Proof. For example, the caracteristic associated to the vector field $-W_3$ is $Q=\omega_u$. Replacing in the relations (37), we obtain $A^1=-\frac{1}{2}\omega_u\omega_v,\ \ A^2=-\frac{1}{2}\omega_u^2$ and thus $P^1=-e^{\omega},\ \ P^2=\frac{1}{2}\omega_u^2$.

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University "Politehnica" of Bucharest
Departament of Mathematics I
Splaiul Independenţei 313
77206 Bucharest Romania
e-mail:nbila@mathem.pub.ro